

A few comments about the Planck-length deformed quantization

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We take a critical view of the following questions concerning the Planck-length deformed quantum mechanics and quantum field theory: 1) The construction of the current vector, the continuity equation and the subsequent challenges for the interpretation of wave-function; 2) The introduction of the coupling between the charged field and an external electromagnetic one, the kinetic term for electromagnetic field and the question of gauge invariance; 3) The classical limit of the Planck-length deformed quantum mechanics and its dramatic consequences.

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I. INTRODUCTION

The Planck-length deformed quantum mechanics as a reasonable model for studying the quantum gravity phenomenology has sparked considerable interest among the physicists [1–6]. In what follows we will use the abbreviation PLQM for the Planck-length deformed quantum mechanics and PLQFT for the Planck-length deformed quantum field theory. Thus far we do not have even a heuristic argument about the equivalence of different representations of PLQM (something like of Stone-von Neumann theorem in standard quantum mechanics [7]). For our discussion we use a particular representation constructed in [8]. This representation looks as follows. We start off with the deformed quantum mechanics

$$\begin{aligned} [\hat{X}^i, \hat{P}^j] &= i \left(\frac{2\beta\hat{\mathbf{P}}^2}{\sqrt{1+4\beta\hat{\mathbf{P}}^2} - 1} \delta^{ij} + 2\beta\hat{P}^i\hat{P}^j \right), \\ [\hat{X}^i, \hat{X}^j] &= [\hat{P}^i, \hat{P}^j] = 0, \end{aligned} \quad (1)$$

where $\beta \propto l_P^2$ is a deformation parameter set by the Planck length: $l_P \approx 10^{-33}$ cm. The deformed $\hat{\mathbf{X}}, \hat{\mathbf{P}}$ operators in Eq.(1) can be represented in terms of the standard $\hat{\mathbf{x}}, \hat{\mathbf{p}}$ operators in the following way

$$\hat{X}^i = \hat{x}^i, \quad \hat{P}^i = \frac{\hat{p}^i}{1 - \beta\hat{\mathbf{p}}^2}. \quad (2)$$

Its Hilbert space realization in the standard-momentum, \mathbf{p} , representation has the form

$$\hat{X}^i\psi(\mathbf{p}) = i\partial_{p_i}\psi(\mathbf{p}), \quad \hat{P}^i\psi(\mathbf{p}) = \frac{p^i}{1 - \beta\mathbf{p}^2}\psi(\mathbf{p}),$$

with the scalar product

$$\langle\psi_1|\psi_2\rangle = \int_{\mathbf{p}^2 < \beta^{-1}} d^3p \psi_1^*(\mathbf{p})\psi_2(\mathbf{p}).$$

Let us notice that the cutoff $\mathbf{p}^2 < \beta^{-1}$ arises merely from the fact that when small p runs over the region $p < \beta^{-1/2}$ large P covers the whole region from 0 to ∞ , see Eq.(2).

It is well known that the familiar Heisenberg uncertainty relation for momentum and position can be viewed as a direct consequence of Fourier analysis. Loosely speaking, the Fourier integral

$$\mathfrak{F}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3p e^{-i\mathbf{p}\cdot\mathbf{x}}\psi(\mathbf{p}),$$

indicates that if the function $\mathfrak{F}(\mathbf{x})$ is localized within the region $\sim r$ then $\psi(\mathbf{p})$ is spread over the region $p \gtrsim r^{-1}$. From this point of view the appearance of the cutoff $\mathbf{p}^2 < \beta^{-1}$ in the above discussion is of crucial importance for having the minimum localization width for the wave-packet of the order of $\sim \beta^{1/2}$. Namely, for this cutoff precludes $\psi(\mathbf{p})$ from being spread over the region greater than $\sim \beta^{-1/2}$, the function $\mathfrak{F}(\mathbf{x})$ cannot be localized beneath the region $\sim \beta^{1/2}$.

II. DOES PLQM VIOLATE THE GAUGE INVARIANCE?

The Lagrangian for a non-relativistic charged particle moving in a given electromagnetic field is given by [9]-§16

$$\mathcal{L} = \frac{m\dot{\mathbf{r}}^2}{2} + q\mathbf{A} \cdot \dot{\mathbf{r}} - q\phi, \quad (3)$$

and hence the Hamiltonian takes the form, $\mathbf{P} = m\dot{\mathbf{r}} + q\mathbf{A}$,

$$\mathcal{H} = \frac{(\mathbf{P} - q\mathbf{A})^2}{2m} + q\phi. \quad (4)$$

The transition to quantum mechanics implies the replacement of \mathbf{P} with the momentum operator [10]-§111. Following this prescription, in the case of PLQM one obtains

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$$\hat{\mathcal{H}} = \frac{(\hat{\mathbf{P}} - q\mathbf{A})^2}{2m} + q\phi = \frac{\hat{\mathbf{P}}^2}{2m} - \frac{q}{2m} (\hat{\mathbf{P}} \cdot \mathbf{A} + \mathbf{A} \cdot \hat{\mathbf{P}}) + \frac{q^2 \mathbf{A}^2}{2m} + q\phi. \quad (5)$$

Let us now derive the quantum mechanical expression for the electric current. It can be derived via the formula

$$\delta\langle H \rangle_\Psi = - \int d^3x \delta\mathbf{A} \cdot \mathbf{J}, \quad (6)$$

where $\delta\langle H \rangle_\Psi$ stands for the variation of $\langle \Psi | \hat{\mathcal{H}} | \Psi \rangle$ caused by the variation $\delta\mathbf{A}$, see [10]-§115. The variation of Eq.(5) with respect to $\delta\mathbf{A}$ amounts to the equation

$$\delta\langle \Psi | \hat{\mathcal{H}} | \Psi \rangle = \int d^3x \Psi^* \left[-\frac{q(\hat{\mathbf{P}} \cdot \delta\mathbf{A} + \delta\mathbf{A} \cdot \hat{\mathbf{P}})}{2m} + \frac{q^2 \mathbf{A} \cdot \delta\mathbf{A}}{m} \right] \Psi,$$

which after using the equality

$$\int d^3x \Psi^* \hat{\mathbf{P}} \cdot \delta\mathbf{A} \Psi = - \int d^3x \delta\mathbf{A} \cdot \Psi \hat{\mathbf{P}} \Psi^*,$$

(that can easily be obtained by using repeated integration and taking into account that each time the surface integral at spatial infinity vanishes) reduces to

$$\delta\langle \Psi | \hat{\mathcal{H}} | \Psi \rangle = \int d^3x \frac{q^2 \mathbf{A} \cdot \delta\mathbf{A}}{m} \Psi^* \Psi + \int d^3x \frac{q\delta\mathbf{A} \cdot (\Psi \hat{\mathbf{P}} \Psi^* - \Psi^* \hat{\mathbf{P}} \Psi)}{2m}. \quad (7)$$

In absence of the background field, from Eq.(7) one finds the current density in the form

$$\mathbf{J} = \frac{\Psi^* \hat{\mathbf{P}} \Psi - \Psi \hat{\mathbf{P}} \Psi^*}{2m} = \frac{i}{2m} \left(\Psi \frac{\nabla}{1 + \beta\Delta} \Psi^* - \Psi^* \frac{\nabla}{1 + \beta\Delta} \Psi \right), \quad (8)$$

which clearly indicates that the continuity equation

$$\frac{\partial(\Psi^* \Psi)}{\partial t} + \text{div} \mathbf{J} = 0,$$

does not hold any more.

The expression of current that would satisfy the continuity equation immediately follows from the equation

$$2mi\partial_t(\Psi^* \Psi) = \Psi^* \hat{\mathbf{P}}^2 \Psi - \Psi \hat{\mathbf{P}}^2 \Psi^* = \sum_{n=0}^{\infty} (1+n)\beta^n \left[\Psi^* (-\Delta)^{(n+1)} \Psi - \Psi (-\Delta)^{(n+1)} \Psi^* \right]. \quad (9)$$

Using the partial-differentiation

$$\begin{aligned} \Psi^* \partial^2 \Psi &= \partial(\Psi^* \partial \Psi) - \partial \Psi^* \partial \Psi, \\ \Psi^* \partial^4 \Psi &= \partial(\Psi^* \partial^3 \Psi) - \partial \Psi^* \partial^3 \Psi = \\ &\partial(\Psi^* \partial^3 \Psi) - \partial(\partial \Psi^* \partial^2 \Psi) + \partial^2 \Psi^* \partial^2 \Psi, \\ \Psi^* \partial^6 \Psi &= \partial(\Psi^* \partial^5 \Psi) - \partial \Psi^* \partial^5 \Psi = \\ &\partial(\Psi^* \partial^5 \Psi) - \partial(\partial \Psi^* \partial^4 \Psi) + \partial^2 \Psi^* \partial^4 \Psi = \\ &\partial(\Psi^* \partial^5 \Psi) - \partial(\partial \Psi^* \partial^4 \Psi) + \partial(\partial^2 \Psi^* \partial^3 \Psi) - \partial^3 \Psi^* \partial^3 \Psi. \end{aligned}$$

one finds

$$\begin{aligned} \sum_{n=0}^{\infty} (1+n)\beta^n \left[\Psi^* (-\Delta)^{(n+1)} \Psi - \Psi (-\Delta)^{(n+1)} \Psi^* \right] &= \text{div} \left[\sum_{n=0}^{\infty} (1+n)\beta^n (-1)^{n+1} (\Psi^* \nabla^{2n+1} \Psi - \Psi \nabla^{2n+1} \Psi^*) + \right. \\ &\sum_{n=1}^{\infty} (1+n)\beta^n (-1)^{n+1} (\nabla \Psi^* \nabla^{2n} \Psi - \nabla \Psi \nabla^{2n} \Psi^*) + \sum_{n=2}^{\infty} (1+n)\beta^n (-1)^{n+1} (\nabla^2 \Psi^* \nabla^{2n-1} \Psi - \nabla^2 \Psi \nabla^{2n-1} \Psi^*) + \\ &\left. \sum_{n=3}^{\infty} (1+n)\beta^n (-1)^{n+1} (\nabla^3 \Psi^* \nabla^{2n-2} \Psi - \nabla^3 \Psi \nabla^{2n-2} \Psi^*) + \dots \right]. \quad (10) \end{aligned}$$

Let us notice that this sort of current (truncated to first order in β) is used in [11].

Another interesting point is the gauge invariance. In

the framework of quantum mechanics, the gauge transformation $\mathbf{A} \rightarrow \mathbf{A} + \nabla f$, $\phi \rightarrow \phi - \partial_t f$ in the Schrödinger equation is compensated by the wave-function transfor-

mation $\Psi \rightarrow \Psi \exp(iqf)$. But now this sort of gauge invariance is violated. It implies that the vector potential has to be seen as a real physical field in PLQM [12]. This point will be addressed in more detail in the following section.

III. INTERACTION OF CHARGED FIELD WITH THE ELECTROMAGNETIC ONE

The action for a charged scalar field in the framework of PLQFT can be written in the form

$$\mathcal{W}[\Phi] = - \int d^4x \frac{1}{2} \left[\hat{P}_\mu \Phi \hat{P}^\mu \Phi^* + m^2 \Phi \Phi^* \right] - \int d^4x \frac{1}{2} \left[\hat{P}_0 \Phi \hat{P}^0 \Phi^* - \hat{\mathbf{P}} \Phi \cdot \hat{\mathbf{P}} \Phi^* + m^2 \Phi \Phi^* \right], \quad (11)$$

where $\hat{P}_0 = i\partial_t$. That is, $(\mathfrak{D}_t \equiv \partial_t, \mathfrak{D}_i \equiv \partial_i(1 + \beta\Delta)^{-1})$

$$\begin{aligned} \mathcal{W}[\Phi] &= \int d^4x \frac{1}{2} \left[\partial_t \Phi \partial_t \Phi^* - \frac{\partial_j}{1 + \beta\Delta} \Phi \frac{\partial_j}{1 + \beta\Delta} \Phi^* - m^2 \Phi \Phi^* \right] \\ &\equiv \int d^4x \frac{1}{2} \left[\mathfrak{D}_\mu \Phi \mathfrak{D}^\mu \Phi^* - m^2 \Phi \Phi^* \right], \end{aligned} \quad (12)$$

which results in the equation of motion

$$\partial_t^2 \Phi - \frac{\Delta}{(1 + \beta\Delta)^2} \Phi + m^2 \Phi \equiv \mathfrak{D}_\mu \mathfrak{D}^\mu \Phi + m^2 \Phi = 0.$$

A. PLQFT - Noether current

Clearly, the Lagrangian has the $U(1)$ symmetry $\Phi \rightarrow e^{i\alpha} \Phi$, $\Phi^* \rightarrow e^{-i\alpha} \Phi^*$, which (in the standard case) is connected with the conservation of electromagnetic current. Following the reasoning of Noether's theorem one finds ($\Phi_1 \equiv \Phi$, $\Phi_2 \equiv \Phi^*$)

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \alpha} = 0 &= \frac{\partial \mathcal{L}}{\partial \Phi_k} \frac{\partial \Phi_k}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial (\mathfrak{D}_\mu \Phi_k)} \frac{\partial (\mathfrak{D}_\mu \Phi_k)}{\partial \alpha} = \\ &= \frac{\partial \mathcal{L}}{\partial \Phi_k} \frac{\partial \Phi_k}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial (\mathfrak{D}_\mu \Phi_k)} \mathfrak{D}_\mu \frac{\partial \Phi_k}{\partial \alpha}. \end{aligned} \quad (13)$$

Using here the equation of motion

$$\frac{\partial \mathcal{L}}{\partial \Phi_k} = \mathfrak{D}_\mu \frac{\partial \mathcal{L}}{\partial (\mathfrak{D}_\mu \Phi_k)},$$

from Eq.(13) one gets

$$\left(\mathfrak{D}_\mu \frac{\partial \mathcal{L}}{\partial (\mathfrak{D}_\mu \Phi_k)} \right) \frac{\partial \Phi_k}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial (\mathfrak{D}_\mu \Phi_k)} \mathfrak{D}_\mu \frac{\partial \Phi_k}{\partial \alpha} = i (\mathfrak{D}_\mu \mathfrak{D}^\mu \Phi^*) \Phi - i (\mathfrak{D}_\mu \mathfrak{D}^\mu \Phi) \Phi^* + i \mathfrak{D}^\mu \Phi^* \mathfrak{D}_\mu \Phi - i \mathfrak{D}^\mu \Phi \mathfrak{D}_\mu \Phi^* = 0. \quad (14)$$

The equation (14) reduces to

$$\begin{aligned} \partial_\mu [i (\partial^\mu \Phi^*) \Phi - i \Phi^* \partial^\mu \Phi] - i \sum_{n=1}^{\infty} (1+n) \beta^n \left[\Phi^* (-\Delta)^{(n+1)} \Phi - \Phi (-\Delta)^{(n+1)} \Phi^* \right] - i \partial_i \Phi^* \sum_{l=1}^{\infty} \partial_i (-\beta\Delta)^l \Phi + \\ i \sum_{n=1}^{\infty} \partial_i (-\beta\Delta)^n \Phi \partial_i \Phi^* - i \sum_{n=1}^{\infty} \partial_i (-\beta\Delta)^n \Phi^* \sum_{l=1}^{\infty} \partial_i (-\beta\Delta)^l \Phi + i \sum_{n=1}^{\infty} \partial_i (-\beta\Delta)^n \Phi \sum_{l=1}^{\infty} \partial_i (-\beta\Delta)^l \Phi^* = 0. \end{aligned}$$

To the first order in β this relation takes the form

$$\begin{aligned} \partial_\mu [i (\partial^\mu \Phi^*) \Phi - i \Phi^* \partial^\mu \Phi] - i\beta [2\Phi^* \Delta^2 \Phi - 2\Phi \Delta^2 \Phi^* - \partial_i \Phi^* \partial_i \Delta \Phi + \partial_i \Delta \Phi \partial_i \Phi^*] = \\ \partial_\mu [i (\partial^\mu \Phi^*) \Phi - i \Phi^* \partial^\mu \Phi] - i\beta \partial_j [2\Phi^* \partial_j \Delta \Phi - 2\Phi \partial_j \Delta \Phi^* - 3\partial_j \Phi^* \Delta \Phi + \Delta \Phi \partial_j \Phi^*] = 0. \end{aligned}$$

So that the zero component of the current will have the standard form

$$J^0 = i\Phi^* \partial^0 \Phi - i (\partial^0 \Phi^*) \Phi, \quad (15)$$

while J^i gets modified as

$$J^j = i\Phi^* \partial^j \Phi - i (\partial^j \Phi^*) \Phi - i\beta [2\Phi^* \partial^j \Delta \Phi - 2\Phi \partial^j \Delta \Phi^* - 3\partial^j \Phi^* \Delta \Phi + \Delta \Phi \partial^j \Phi^*]. \quad (16)$$

B. Coupling to the electromagnetic field

The coupling to an external electromagnetic field can be introduced by replacing P_μ in Eq.(12) with the $P_\mu -$

eA_μ [15]-§32. That means to define "covariant" deriva-

tive as

$$\nabla_\mu = \mathfrak{D}_\mu + ieA_\mu , \quad (17)$$

$$\mathcal{W}[\Phi] = \int d^4x \frac{1}{2} [\mathfrak{D}_\mu \Phi \mathfrak{D}^\mu \Phi^* - m^2 \Phi \Phi^* + ieA_\mu \{\Phi^* \mathfrak{D}^\mu \Phi - (\mathfrak{D}^\mu \Phi^*) \Phi\}] , \quad (18)$$

Let us emphasize that the action (18) is not invariant under the gauge transformation

$$\Phi \rightarrow e^{i\alpha} \Phi , \quad A_\mu \rightarrow A_\mu - \frac{ie^{-i\alpha}}{e} \mathfrak{D}_\mu e^{i\alpha} .$$

This description of the electromagnetic coupling seems to us the most intuitive picture following immediately from Eq.(5) than making the gauging of $U(1)$ symmetry in Eq.(12), that is, replacing ∂_μ in Eq.(12) with $\partial_\mu + ieA_\mu$ [16, 17].

To have an uniform picture, the kinetic term for the electromagnetic field can be introduced merely by replacing: $\partial_\mu \rightarrow \mathfrak{D}_\mu$ in the tensor of the electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow \mathcal{F}_{\mu\nu} = \mathfrak{D}_\mu A_\nu - \mathfrak{D}_\nu A_\mu .$$

This definition of kinetic term goes back to the paper [16]. For yet another approach to the electromagnetic kinetic term see [18]. An alternative path used in paper [18] consists of the following steps. The standard action is taken as a starting point

$$\begin{aligned} \mathcal{W} &= -\frac{1}{4} \int dt d^3x F_{\mu\nu} F^{\mu\nu} = \\ &\frac{1}{4} \int dt d^3x [2F_{0i} F_{0i} - F_{ik} F_{ik}] = \\ &\frac{1}{2} \int dt d^3x [\mathbf{E}^2 - \mathbf{B}^2] . \end{aligned}$$

Then the the gauge conditions $A_0 = \partial_i A_i = 0$ are imposed that results in

$$\mathcal{W} = \frac{1}{2} \int dt d^3x [(\partial_0 \mathbf{A})^2 - (\nabla \times \mathbf{A})^2] .$$

The operator ∇ is identified with $\mathbf{p} = -i\nabla$ and the modification due to momentum deformation is understood as

$$\mathcal{W} = \frac{1}{2} \int dt d^3x [(\partial_0 \mathbf{A})^2 - (\mathbf{P} \times \mathbf{A})^2] .$$

In analogy to the $U(1)$ case, for the Yang-Mills field one can write

$$\mathcal{F}_{\mu\nu} = \mathfrak{D}_\mu A_\nu - \mathfrak{D}_\nu A_\mu + ig[A_\mu, A_\nu] ,$$

$$\mathcal{W} = -\frac{1}{4} \int d^4x \text{Sp} (\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) .$$

The equation of motion can be derived easily in the matrix form

$$\delta \mathcal{W} = -\frac{1}{2} \int d^4x \text{Sp} (\mathcal{F}_{\mu\nu} \delta \mathcal{F}^{\mu\nu}) ,$$

where

$$\delta \mathcal{F}^{\mu\nu} = \mathfrak{D}^\mu \delta A^\nu + ig \delta A^\mu A^\nu + ig A^\mu \delta A^\nu - (\mu \leftrightarrow \nu) .$$

Because the antisymmetry of $\mathcal{F}^{\mu\nu}$

$$\begin{aligned} \delta \mathcal{W} &= \\ &- \int d^4x \text{Sp} \{ \mathcal{F}_{\mu\nu} (\mathfrak{D}^\mu \delta A^\nu + ig \delta A^\mu A^\nu + ig A^\mu \delta A^\nu) \} . \end{aligned}$$

The first term in this equation can be integrated by parts, throwing away the surface terms and using the cyclic properties of the **Spur**, it reduces to

$$\delta \mathcal{W} = \int d^4x \text{Sp} \{ (\mathfrak{D}^\mu \mathcal{F}_{\mu\nu} + ig[A^\mu, \mathcal{F}_{\mu\nu}]) \delta A^\nu \} ,$$

from which we read off the equation of motion in the matrix form

$$\mathfrak{D}^\mu \mathcal{F}_{\mu\nu} + ig[A^\mu, \mathcal{F}_{\mu\nu}] = 0 .$$

If the coupling of Yang-Mills field to the external source is introduced: $\int d^4x \text{Sp} (A^\nu J_\nu)$, then one obtains the equation of motion

$$\mathfrak{D}^\mu \mathcal{F}_{\mu\nu} + ig[A^\mu, \mathcal{F}_{\mu\nu}] = J_\nu .$$

Let us focus on $U(1)$ case; the Maxwell equations get modified as

$$\mathfrak{D}^\mu \mathcal{F}_{\mu\nu} = J_\nu .$$

To see how the Poisson equation gets modified, let us consider a static source: $J_0(\mathbf{x}), J_j = 0$. One obtains the equation

$$\mathfrak{D}^j \mathcal{F}_{j0} = \mathfrak{D}^j \mathfrak{D}_j A_0 = -\frac{\Delta}{(1 + \beta\Delta)^2} A_0 = J_0(\mathbf{x}) . \quad (19)$$

To make a proper analysis of this equation let us recall that the operator $\Delta(1 + \beta\Delta)^{-2}$ arises as a result of

using the deformed momentum operator in field theory (see Eq.(11)). Therefore, the Fourier representation of the field includes the cutoff $\mathbf{p}^2 < \beta^{-1}$ (see the section Introduction)

$$A_0(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{p}^2 < \beta^{-1}} d^3p e^{-i\mathbf{p}\cdot\mathbf{x}} \mathfrak{A}_0(\mathbf{p}) ,$$

and similar cutoff is implied for the current as well

$$J_0(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{p}^2 < \beta^{-1}} d^3p e^{-i\mathbf{p}\cdot\mathbf{x}} \mathfrak{J}_0(\mathbf{p}) .$$

For this reason, one infers that the source J_0 cannot be localized beneath the Planck length (see the section Introduction). The solution of Eq.(19) can be constructed as

$$A_0(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{p}^2 < \beta^{-1}} d^3p e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{(1 - \beta\mathbf{p}^2)^2 \mathfrak{J}_0(\mathbf{p})}{\mathbf{p}^2} .$$

For the source represented by the following cutoff version of the δ function

$$J_0(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{p}^2 < \beta^{-1}} d^3p e^{-i\mathbf{p}\cdot\mathbf{x}} ,$$

the expression for A_0 can be found in [20].

In order for A_μ to have the equation similar to the Eq.(13), one has to require a subsidiary condition $\mathfrak{D}^\mu A_\mu = 0$; then for the free field the equation of motion takes the form $\mathfrak{D}^\mu \mathfrak{D}_\mu A_\nu = 0$. This insures to have similar dispersion relations for scalar and vector particles. But now the condition $\mathfrak{D}^\mu A_\mu = 0$ implies two polarization degrees of freedom instead of four.

IV. CLASSICAL LIMIT FOR PLQM

A. Minimum-length deformed classical dynamics

In PLQM the Hamilton's operator takes the form

$$\hat{\mathcal{H}} = \frac{\hat{\mathbf{P}}^2}{2m} + V(\hat{\mathbf{r}}) = \frac{\hat{\mathbf{p}}^2}{2m(1 - \beta\hat{\mathbf{p}}^2)^2} + V(\hat{\mathbf{r}}) . \quad (20)$$

For the velocity operator one finds¹ [10]

¹ The calculation is easy to do in the \mathbf{p} representation: $\hat{\mathbf{p}} = \mathbf{p}$, $\hat{\mathbf{r}} = i\partial/\partial\mathbf{p}$.

$$\hat{\mathbf{v}} \equiv \dot{\hat{\mathbf{r}}} = i [\hat{\mathcal{H}}, \hat{\mathbf{r}}] = \sum_{n=0}^{\infty} \frac{(1+n)\beta^n}{2m} [\hat{\mathbf{p}}^{2(n+1)}, \hat{\mathbf{r}}] = \frac{\hat{\mathbf{p}}}{m} \sum_{n=0}^{\infty} (1+n)^2 \beta^n \hat{\mathbf{p}}^{2n} = \frac{\hat{\mathbf{p}}}{m} \frac{1 + \beta\hat{\mathbf{p}}^2}{(1 - \beta\hat{\mathbf{p}}^2)^3} .$$

To pass to the classical mechanics one has to replace the commutator with the Poisson bracket in the following way [10]

$$\frac{i}{\hbar} [\hat{f}_1, \hat{f}_2] \rightarrow \{f_1, f_2\} .$$

So we arrive at the Hamilton equations

$$\dot{\mathbf{r}} = \{\mathcal{H}, \mathbf{r}\} , \quad \dot{\mathbf{p}} = \{\mathcal{H}, \mathbf{p}\} ,$$

with the standard Poisson brackets [21]

$$\{x_i, x_k\} = 0 , \quad \{p_i, p_k\} = 0 , \quad \{p_i, x_k\} = \delta_{ik} ,$$

but now in view of Eq.(20) the Hamiltonian is modified as

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m(1 - \beta\mathbf{p}^2)^2} + V(\mathbf{r}) . \quad (21)$$

B. Kepler's problem

In classical dynamics there are only two cases in which all the bounded orbits in central fields are closed, namely, $V(r) = ar^2$, $a > 0$ and $V(r) = -\alpha/r$, $\alpha > 0$. The Hamiltonian for describing the motion in a Newtonian potential gets modified as (see Eq.(21))

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m(1 - \beta\mathbf{p}^2)^2} - \frac{\alpha}{r} = \mathcal{H}_0 + \sum_{n=1}^{\infty} \frac{1+n}{2m} \beta^n \mathbf{p}^{2(n+1)} .$$

The angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is still conserved

$$\frac{d\mathbf{L}}{dt} = \{\mathcal{H}, \mathbf{L}\} = \mathbf{r} \times \{\mathcal{H}, \mathbf{p}\} + \{\mathcal{H}, \mathbf{r}\} \times \mathbf{p} = \mathbf{r} \times \left(-\frac{\alpha}{r^2} \frac{\mathbf{r}}{r}\right) + \frac{d}{dp} \left(\frac{p^2}{2m(1 - \beta p^2)^2}\right) \frac{\mathbf{p}}{p} \times \mathbf{p} = \mathbf{0} .$$

So, the motion occurs in the plane perpendicular to \mathbf{L} . Let x, y be rectangular coordinates in this plane. Then from the equations of motion one gets

$$\begin{aligned}\dot{x} &= \{\mathcal{H}, x\} = \frac{p_x}{m} \frac{1 + \beta p^2}{(1 - \beta p^2)^3}, \\ \dot{y} &= \{\mathcal{H}, y\} = \frac{p_y}{m} \frac{1 + \beta p^2}{(1 - \beta p^2)^3},\end{aligned}$$

and correspondingly

$$\begin{aligned}m\sqrt{\dot{x}^2 + \dot{y}^2} &= p \frac{1 + \beta p^2}{(1 - \beta p^2)^3} = p + 4\beta p^3 + \\ &9\beta^2 p^5 + 16\beta^3 p^7 + 25\beta^4 p^9 + O(\beta^5). \quad (22)\end{aligned}$$

Can we restrict ourselves to the first order in β in discussing the motion of solar system planets? The validity of this approximation means that $\beta m^2 v^2 \ll 1$. Recalling that $\beta \sim m_P^{-2}$, one observes that since the mean orbital velocities of the planets are by a few orders of magnitude smaller than the light velocity in vacuum but their masses are much more greater as compared to m_P , the above condition is not satisfied. Using the mean orbital velocities and masses of planets one gets that the order of magnitude for $\beta m^2 v^2$ varies from 10^{54} to 10^{64} . For the motion of moon around the earth this quantity is of the order of 10^{47} .

Let us focus on the motion of the moon in earth's gravitational field. The energy expression (which is certainly a conserved quantity) takes the form

$$E = \frac{\mathbf{P}^2}{2M_{moon}} - \frac{m_P^{-2} M_{earth} M_{moon}}{r}, \quad (23)$$

where $M_{earth} = 5.9736 \times 10^{24}$ kg and $M_{moon} = 0.07349 \times 10^{24}$ kg [22]. Using $m_P \approx 2.17651 \times 10^{-8}$ kg one finds $m_P^{-2} M_{earth} M_{moon} \approx 0.2017 \times 10^{40}$. The orbital velocities of the moon at the perigee $r_+ = 363300$ km and apogee $r_- = 405500$ km are $v_+^{(0)} = 1.076$ km/s and $v_-^{(0)} = 0.964$ km/s respectively [22] (we introduced the notation $v^{(0)}$ to distinct between standard and modified cases). From Eq.(22) one observes that for this range of velocities p^2 is close to $1/\beta$ with a great accuracy. This fact allows to somewhat simplify the Eq.(22)

$$mv = p \frac{1 + \beta p^2}{(1 - \beta p^2)^3} = P^3 \left(\beta + \frac{1}{p^2} \right) \approx 2\beta P^3. \quad (24)$$

Now taking $v_- = v_-^{(0)}$ and $\beta = m_P^{-2}$, from Eqs.(23, 24) one finds

$$\begin{aligned}\left(\frac{v_+}{2}\right)^{2/3} &= \left(\frac{v_-^{(0)}}{2}\right)^{2/3} + \\ &\left(\frac{M_{moon}}{m_P}\right)^{4/3} \left(\left[v_+^{(0)}\right]^2 - \left[v_-^{(0)}\right]^2 \right). \quad (25)\end{aligned}$$

Substituting the above cited quantities in Eq.(25) one finds (in natural units)

$$v_+ \simeq 10^{43}.$$

V. DISCUSSION

We address a number of questions concerning the broad class of PLQM admitting Hilbert space representation in which the deformation is completely ascribed to the momentum operator [23].

First we ask how to introduce electromagnetic interaction in PLQM. To answer this question, we take into account that the standard rule for inclusion of electromagnetism in quantum mechanics $\nabla \rightarrow \nabla + iq\mathbf{A}$ comes from the fact that classically the coupling of the electromagnetism to the particle leads to the replacement $\mathbf{P} \rightarrow \mathbf{P} + q\mathbf{A}$; see Eqs.(3, 4). Taking the same classical picture as a starting point, one notices that in the case of PLQM the transition to quantum formalism is achieved with the transcription $\mathbf{P} \rightarrow -i\nabla/(1 + \beta\Delta)$ and therefore an interaction with an external electromagnetic field is introduced by the substitution $\nabla/(1 + \beta\Delta) \rightarrow \nabla/(1 + \beta\Delta) + iq\mathbf{A}$; see Eqs.(2, 5). That is, we uniquely follow the way: $\hat{\mathbf{P}} \rightarrow \hat{\mathbf{P}} + q\mathbf{A}$. Generalization of this rule to the case of four-potential (A^0, \mathbf{A}) is straightforward as the operator \hat{P}^0 is unmodified; see Eqs.(11, 12, 17). Let us notice that our approach differs from that one suggested in [16, 17] to use the rule $\partial_\mu \rightarrow \partial_\mu + iqA_\mu$ for all ∂_μ operators in the deformed momentum. The proposal of [16, 17] leads to the infinite number of interaction terms, that is, the interaction term is represented as a series of powers of β . In our case it is easy to see that one arrives at a finite number of interaction terms, see Eq.(18).

Next, we ask how to derive the expression of electric current in PLQM. Following the Landau-Lifshitz derivation, see Eq.(6), the electric current is defined by means of the background electromagnetic field \mathbf{A} as

$$\frac{\delta \langle H \rangle_\Psi}{\delta \mathbf{A}(\mathbf{x})} = \mathbf{J}(\mathbf{x}).$$

But the current derived this way does not satisfy the continuity equation and therefore the standard probabilistic interpretation of the wave-function becomes obscure, Eq.(8). On the other hand, one can define the conserved current immediately from the Schrödinger equation, but then its physical interpretation becomes obscure, Eqs.(9, 10). Similarly, in the case of PLQFT Noether's current, see Eqs.(15, 16), does not coincide with the one that comes immediately from the action functional after inclusion of the electromagnetic field, Eq.(18).

Another interesting question is the gauge invariance; the above discussion manifests the violation of gauge invariance in PLQM, PLQFT (last paragraph in section II and the Eq.(18)). Therefore, the degrees of freedom that

are reduced due to gauge invariance in the standard case are now physical. But, nevertheless, in order to have the same dispersion relation for all particles, one has to impose on the electromagnetic field the additional condition (last paragraph in section III).

Finally (section IV) we point out dangerous implications of PLQM for classical physics. Namely, its classical counterpart manifests huge effects when applied for the motion of planets. Similar observations were made in [24] in the framework of different Hilbert space represen-

tation of PLQM and also in [25]. The latter paper used the above discussed representation but consideration was restricted to the first order in β that for our purposes can not be considered a good approximation.

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